Design of FIR filter for DAC distortion compensation

*Mahdi A. Hafi (m.hafi@uot.edu.ly)

ABSTRACT

Digital signal processing (DSP) has been and is still a major block in the design of many system applications. It is also involved in the manufacturing of equipment/devices in the fields of digital communications, digital control, ...etc. (Steven W. Smith, Robert Oshana 2012). In our physical world many signals are analog by nature, and this necessitates conversion process from one type to another. Conversion however, can lead to some information loss or may not be preserved. It can also result in signal distortion(James H. McClellan, et al). Depending on the accuracy and sensitivity of the information, extra steps maybe taken in the overall system design to compensate for the distortion and/or information loss. This paper will deal with an approach that would minimize this distortion through the design and implementation of FIR type digital filter.

KEYWORDS: Digital-to-analog converter (DAC), Digital filter, FIR filter, Low-pass filter (LPF), Inverse Discrete Fourier Transform (IDFT), and discrete impulse response.

INTRODUCTION

When a band-limited signal is sampled at the rate required (Fig. 1) " $T \leq \pi/\omega_c$ where ω_c is the cutoff frequency of the signal", then its discrete time Fourier transform is a periodic version with period of 2π, that is, (Alan V. Oppenheim,et al 1975):

*Staff member faculty of engineering university of Tripoli-Libya

$$X\left(e^{j\omega T}\right)=\frac{1}{T}\sum_{r=-\infty}^{+\infty} X_a\left(j\omega+j\frac{2\pi r}{T}\right) \qquad (1)$$

where $X_a(j\omega)$ is the Fourier transform of the analog signal as shown by Fig. 2.

As can be seen from the same figure, if an analog low–pass filter with cutoff requency of ω_c is applied to the sampled signal $x(nT)$, then an exact recovery of the original analog signal $x_a(t)$ is possible. True, if the sampled signal was somehow continuous in time, and since it is not, an in–between step has to take place.

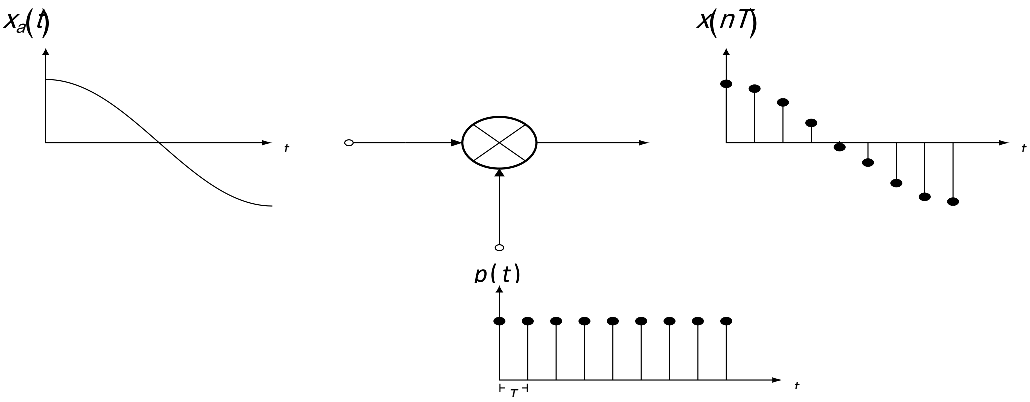


Figure 1: Signal $x_a(t)$ and its samples $x(nT)$

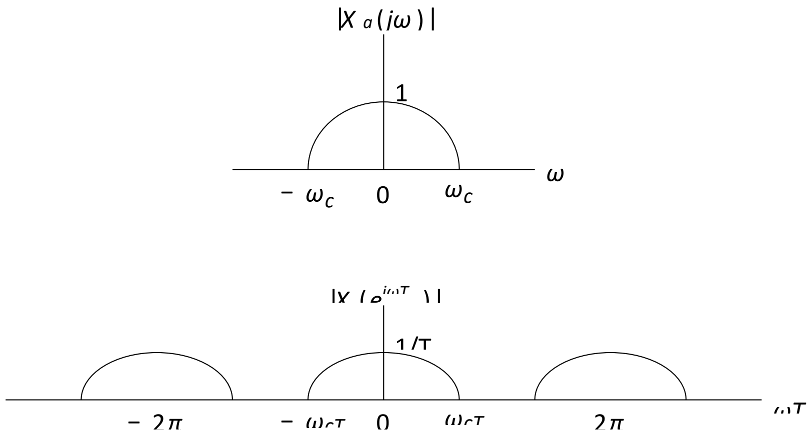


Figure 2: Magnitude spectrum of signal $X_a(j\omega)$ and its sampled version $X(e^{j\omega T})$

Using digital–to–analog converter (DAC) as shown by Fig. 3 is usually the way to convert from discrete time signal to continuous time signal, and in doing so and as a result, the Fourier transform of its staircase output signal is not the same as that of the input $x(nT)$, (William D. Stanley. 1992). In an attempt to recover the original signal with minimum error, another filter characteristics other than that of a low–pass should be used in order to compensate for the “frequency response distortion” introduced by DAC.

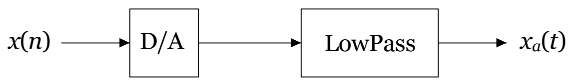


Figure 3: Typical conversion process of digital signal to analog signal.

DESIGN OF A DIGITAL FILTER

Fig. 4 shows a block diagram of an analog signal to be recovered from its samples using, digital filter, DAC, and a low–pass filter. Here, the digital filter is to compensate for the “frequency response distortion introduced by DAC”. In order to derive the characteristic of the digital filter, the original signal is assumed to be band–limited with a cutoff frequency ω_c and sampled at the required rate T. Knowing that: $X\left(e^{j\omega t}\right)=\frac{1}{T}X_a(j\omega)$ for $-\frac{\pi}{T}\leq\omega\leq\frac{\pi}{T}$ and $x_a(t=nT)=x(nT)$ where n is an integer, then from the block diagram shown in Fig. 4,

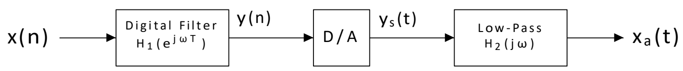


Figure 4: Adding digital filter to the conversion process from digital to analog signal.

$$X_a(j\omega)=H_2(j\omega)Y_s(j\omega) \qquad for \qquad |\omega|\leq\omega_c$$

where

$$H_2(j\omega)=\begin{cases}1, & |\omega|\leq\omega_c\\ 0, & otherwise\end{cases}$$

This implies that

$$Y_s(j\omega)=X_a(j\omega) \qquad for \qquad |\omega|\leq\omega_c \tag{2}$$

Since $y_s(t) = y(nT)$ for $nT \le t \le (n+1)T$ then,

$$
\begin{aligned}
Y_s(j\omega) &= \int_{-\infty}^{+\infty} y_s(t) e^{-j\omega t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} y(nT) e^{-j\omega t} dt \\
&= \sum_{n=-\infty}^{\infty} y(nT) \int_{nT}^{(n+1)T} e^{-j\omega t} dt \\
&= T\, sinc\left(\frac{\omega T}{2}\right) e^{-j\omega T/2} \sum_{n=-\infty}^{\infty} y(nT) e^{-jn\omega T}
\end{aligned}
$$

since $Y\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} y(nT) e^{-jn\omega T}$, then

$$Y_s(j\omega) = T\, sinc\left(\omega \tfrac{T}{2}\right) e^{-j\omega \frac{T}{2}}\, Y\left(e^{j\omega T}\right) \qquad (3)$$

from Eq. (1),

$$X_a(j\omega) = T\, sinc\left(\omega \tfrac{T}{2}\right) e^{-j\omega \frac{T}{2}}\, Y\left(e^{j\omega T}\right) \quad for \quad |\omega| \le \omega_c$$

Also, from the block diagram in Fig. 4, $Y\left(e^{j\omega T}\right) = H_1\left(e^{j\omega T}\right) X\left(e^{j\omega T}\right)$, which leads to

$$X_a(j\omega) = T\, sinc\left(\omega \tfrac{T}{2}\right) e^{-j\omega \frac{T}{2}}\, H_1\left(e^{j\omega T}\right) X\left(e^{j\omega T}\right) \quad for \quad |\omega| \le \omega_c$$

We also have from Eq. (1), $X\left(e^{j\omega T}\right) = \frac{1}{T} X_a(j\omega)$ for $|\omega| \le \omega_c$ and so

$$\boxed{H_1\left(e^{j\omega T}\right) = \frac{e^{j\omega \frac{T}{2}}}{sinc\left(\omega \frac{T}{2}\right)}} \qquad (4)$$

For $|\omega| \le \omega_c$. Note that the term $\left(T\, sinc\left(\omega \frac{T}{2}\right) e^{-j\omega \frac{T}{2}}\right)$ in Eq. (3) is the frequency response distortion introduced by the DAC. If one were to investigate the digital filter transfer function derived in Eq. (4), he would find that it represents one period of a continuous periodic spectrum in 2π of a sampled signal $x_a(t)$. If, however, the frequency response in Eq. (4) is treated as non−periodic, then the signal can be considered as continuous and its inverse Fourier transform will yield the impulse time response $h(t)$. The inverse Fourier transform is given by

$$h(t) = \int_{-\omega_c}^{\omega_c} \frac{\omega e^{j\frac{\omega}{2}}}{2sinc\left(\frac{\omega}{2}\right)}\, e^{j\omega t} d\omega$$

and it is indefinite. This means that finding the transfer function will not be in simple form and for us this rule out the implementation and design of an IIR filter. For this particular issue then, two sought design methods based on FIR filter implementation are discussed. Since the transfer function in Eq. (4) has even magnitude and odd phase response as shown by Fig. 5, the coefficients of $h(n)$ are then real(Gabel, & Roberts.1980). To determine the finite sequence of $h(n)$, of the FIR filter, two methods based on frequency sampling realization are considered.

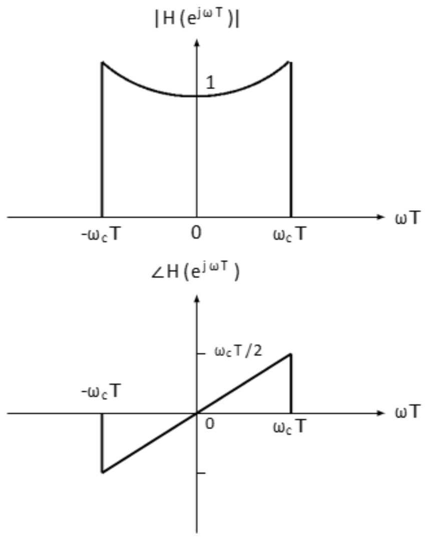


Figure 5: Magnitude and phase plot of $H\left(e^{j\omega T}\right) = e^{j\omega\frac{T}{2}}/sinc(\omega\frac{T}{2})$ for $|\omega T| \leq \pi$

1− Sampling of $H(z)$ method

It can be shown that for any sampled signal, its periodic frequency spectrum can be related to its frequency samples "$H(k)$" and for N number of samples that occur every $\omega_k = \frac{2\pi k}{NT}$, (Alan V. Oppenheim,et al 1975), by

$$H(e^{j\omega T}) = \left(1 - e^{-j\omega TN}\right)\frac{1}{N}\sum_{k=0}^{N-1}\frac{\widetilde{H}(k)}{1 - e^{j\frac{2\pi k}{NT}}e^{j\omega T}} \tag{5}$$

With the fact that $h(n)$ is real, that is $|H(k)| = |H(N-k)|$ and $\theta(k) = \theta(N-K)$, then Eq. (5) can be rewritten for N even as

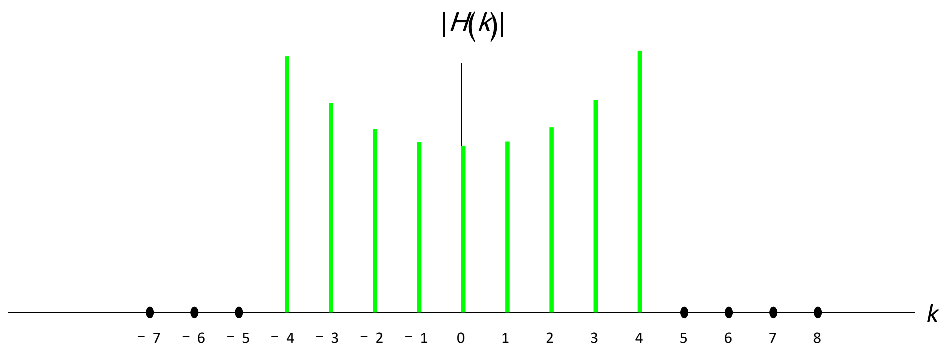
$$H(e^{j\omega T}) = \left(1 - e^{-j\omega TN}\right)\left\{\sum_{k=1}^{\frac{N}{2}-1}\frac{2\left|\widetilde{H}(k)\right|}{N}\, H_k\left(e^{j\omega T}\right) + \frac{\widetilde{H}(0)}{1 - e^{-j\omega T}} - \frac{\widetilde{H}\left(\frac{N}{2}\right)}{1 - e^{-j\omega T}}\right\} \tag{6}$$

where

$$H_k\left(e^{j\omega T}\right) = \frac{\cos(\theta(k)) - \left[\cos\left(\theta(k) - \frac{2\pi k}{N}\right)\right] e^{-j\omega T}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) + e^{-j2\omega T}}$$

The filter magnitude and phase plots are shown in Fig. 6, and it's realization is shown in Fig. 7 for N = 16. Where

$$T = \cos\left(\theta(k) - \frac{2\pi k}{N}\right),\ \ S = \cos\left(\theta(k)\right), \text{and}\ \ G = 2\cos\left(\frac{2\pi k}{N}\right).$$



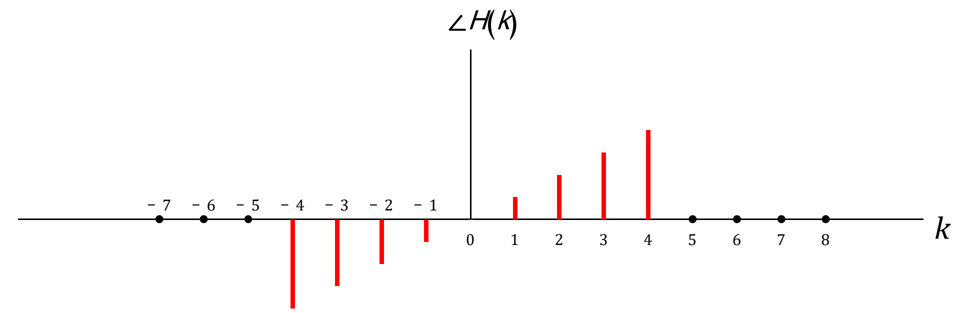


Figure 6: Magnitude and phase of $H(k) = H(e^{j\omega T})|_{\omega=\frac{2\pi k}{NT}}$, and $N = 16$

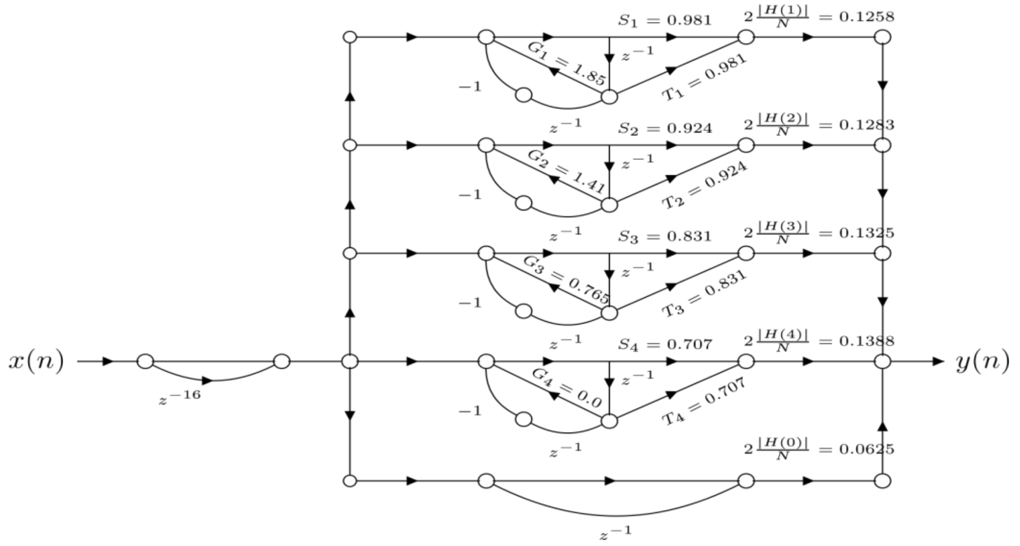


Figure 7: FIR filter diagram with (N=16)

2− Using IDFT method

Since the frequency samples "$H(k)$" can be obtained simply by the relation $H(k) = H(e^{j\omega T})|_{\omega_k=\frac{2\pi k}{NT}}$, then the coefficients of $h(n)$ can be determined by using the inverse discrete Fourier transform "IDFT" i.e.,

$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi kn}{N}}$ where $k = 0,1,2,\ldots,N-1$, and N is the number of samples. For this particular filter then,

$$
\begin{aligned}
h(n) &= \frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{j\frac{\pi k}{N}}}{sinc\left(\frac{\pi k}{N}\right)}\, e^{j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}\sum_{\omega_k=-\omega_c}^{\omega_c}\frac{e^{j\frac{\omega_k T}{2}}}{sinc\left(\frac{\pi k}{N}\right)}\, e^{jn\omega_k T}
\end{aligned}
$$

where $\omega_k = \frac{2\pi k}{NT}$, and $k = 0,1,2,\dots,N-1$.

Once the filter coefficients are determined, its output $y(n)$ is related to its input $x(n)$ by the relation $y(n) = x(n) * h(n) = \sum_{k=0}^{N-1} h(k)x(n-k)$. The realization of such relation is called the Direct–Form as show by Fig. 8.

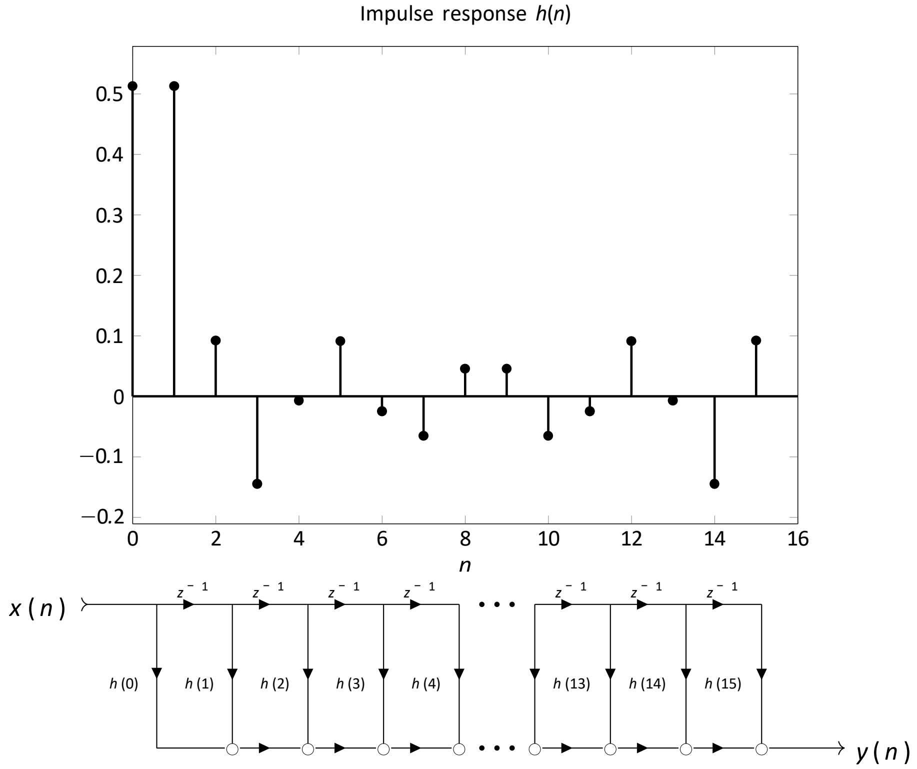


Figure 8: FIR filter Direct–Form realization (N=16)

Table 1: FIR filter $h(n)$ coefficients (N=16)

n	$h(n)$	n	$h(n)$
0	00.5128	8	0.0456
1	0.5128	9	0.0456
2	0.0922	10	−0.0653
3	−0.1449	11	−0.0247
4	−0.0069	12	0.0913
5	0.0913	13	−0.0069
6	−0.0247	14	−0.1449
7	−0.0653	15	0.0922

Tests & Observations:

Using MATLAB(Roberts, M.J. 2012& DIMITRIS et al), the following filter was implemented as shown in the network of Fig. 9. Selecting Direct–Form, and from a practical point of view, with specification of (ω_c = 1*k rad/s, N* = 16, and $T = \pi/2\omega_c$). The system is subjected to a band–limited signal input $x(m)$. It is chosen to be of the form $x(m) = \sin(\omega_o mT)/\pi mT$, where ω_o is the cutoff frequency of the input signal, M the number of input samples, and T is the sampling period. The input signal and its frequency response are plotted as shown by Fig. 10 where ω_o is chosen to be 500 *rad/s* (a value less than that of the filter ω_c = 1*k rad/s*) and T to be the same as that of the filter ($T = \pi/2\omega_c$) since the later has a larger cutoff frequency. The output results of the system are almost as expected and as shown by the plots. In general, the larger the number of samples M *i.e.*, (reducing the truncation error), the closer is the frequency response to that of the analog signal $X_a(j\omega)$. Note that for a reasonable number of samples such as M = 16, the truncation error is small in a way that the magnitude frequency response of $X(e^{j\omega T})$ is much like the actual one $X_a(j\omega)$. Note also, that the filter itself is truncated and the larger the number of *n,* the better would be the response, but from practical point of view that one should have such a filter that is as satisfactory and as simple as possible. The previous results dealt with the case of band–limited signal, but in real life applications, this is not the case and so it would be proper to test for signals that are not band–limited. Choosing $x(m)$ to be of the form; $x(m) = \omega_o\, e^{\omega_o mT}$, where ω_o is the upper 3db frequency and T is the same as that of the filter for the same reason mentioned earlier $(T = \pi/2\omega_c)$. Since the signal is not band–limited, ω_o is chosen to be much smaller than that of the of the filter for better results. ω_o is chosen to be ten times less ($\omega_o = 100$ *rad/s*). Here, again there is truncation error and also aliasing error due to the fact that the signal is not band–limited. It is worth noticing and as shown by Fig. 11, that for M=*16*, the shape of the magnitude response of $X(e^{j\omega T})$ follows almost exactly as that of $X_a(j\omega)$ for at least a wide range of frequencies. One can conclude then that for a reasonable number of samples ($M = 16$), the errors are still small and the overall response is pretty much like that of the original analog signal $x_a(t)$.

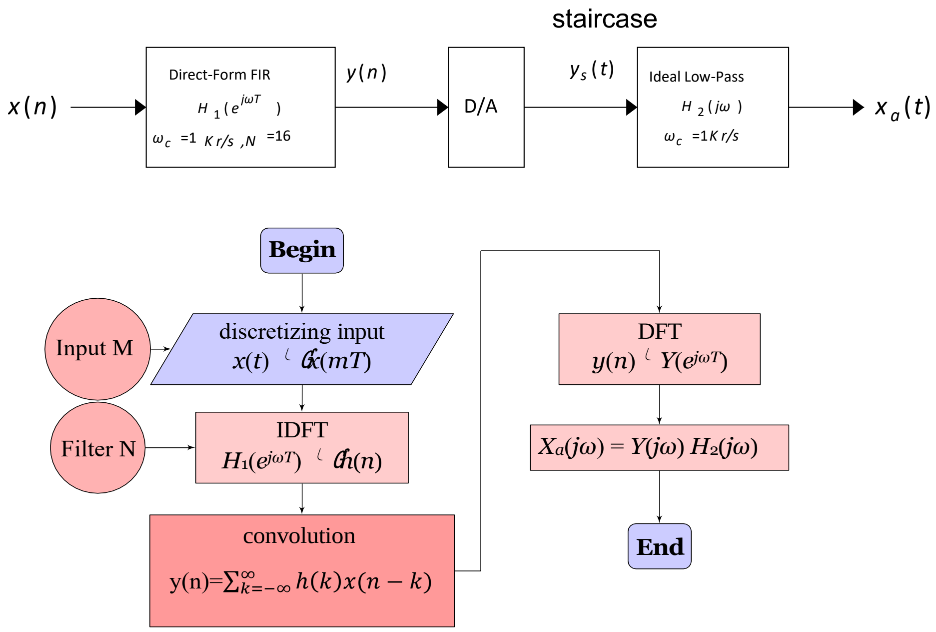


Fig. 9: Block diagram of the system realization and its implementation flow chart

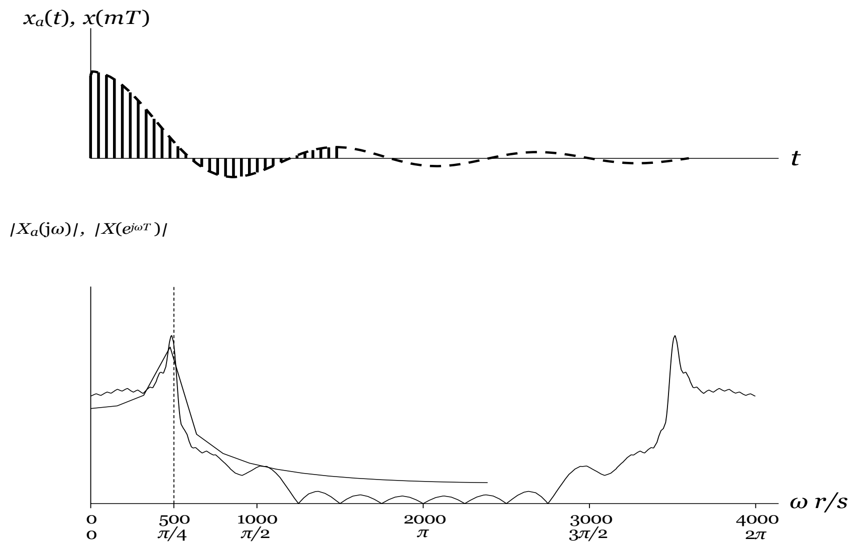


Fig. 10: signals $x_a(t)$, $x(mT)$, $/X_a(j\omega)/$, and $/X(e^{j\omega T})/$

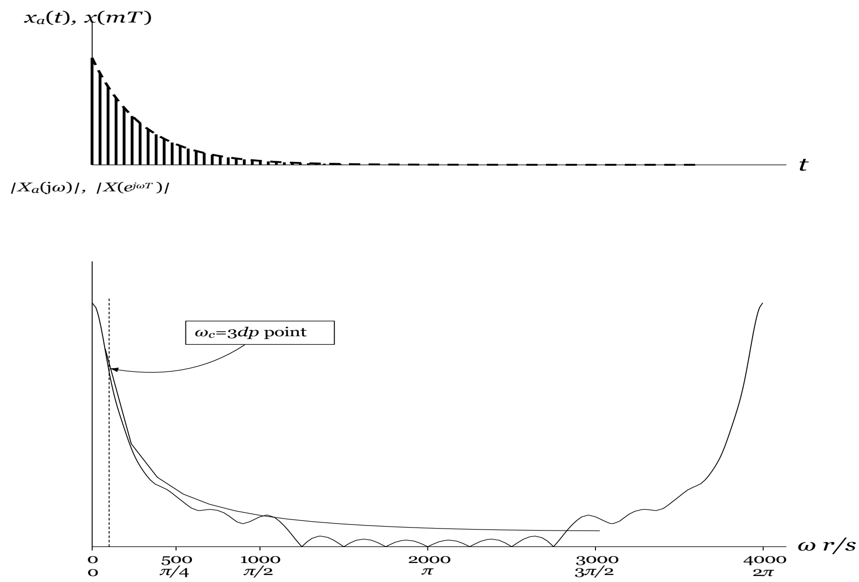


Fig. 11: signals $x_a(t)$, $x(mT)$, $/X_a(j\omega)/$, and $/X(e^{j\omega T})/$

CONCLUSION

Digital filter (FIR type) was designed to compensate for the distortion caused by Digital-to-Analog converters. Two sample input signals were tested. One was band-limited signal, and the other was not. In either case the output of the system, which is the converted back analog signal, was successfully recovered with minor discrepancies that are caused by truncation and aliasing errors.

References:

[1] Steven W. Smith. *The Scientist and Engineer's Guide to Digital Signal Processing* (Second Edition). California Technical Publishing, San Diego, California.

[2] Robert Oshana. *DSP for Embedded and Real-Time Systems*. Elsevier Inc., Waltham, MA, USA, 2012.

[3] James H. McClellan, Ronald Schafer, Mark Yoder. *DSP First* (Second Edition) Pearson Education, Inc., Hoboken, New Jersey.

[4] Alan V. Oppenheim, Ronald W. Schafer. *Digital Signal Processing*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.

[5] William D. Stanley. *Electronic Communications Systems*. Reston Publishing Company, Inc., Reston, Virginia, 1992.

[6] Robert A. Gabel, Richard A. Roberts. *SIGNALS AND LINEAR SYSTEMS, Second Edition*. John Wiley and Sons, Inc., New York, New York, 1973 and 1980.

[7] Roberts, M.J. (2012). *Signals and Systems: Analysis Using Transform Methods and MATLAB ®(2nd ed.)*. New York, NY: McGraw-Hill.

DIMITRIS G. MANOLAKIS, VINAY K. INGLE. *Applied Digital Signal Processing*. Cambridge University Press, New York